

UNITED STATES MILITARY ACADEMY
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An Introduction to Estimation and Comparison of Discrete Variate Time Processes

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Introduction: What is a Time Series?

- A collection of random variables indexed according to the order they are obtained in time (equally spaced).
  - Years, months, weeks, days, hours, minutes, seconds, etc.
  - Examples:
    - Economics: stock market, monthly unemployment
    - Social Science: crime data, school enrollment
    - Biology: number of people with a certain disease
    - Environmental Scientists: temperature, air quality
    - Business: revenue, sales, customers
- One major difference is that time series analysis violates the independence assumption common in other statistics fields (such as linear regression).
- Autocorrelation is the idea that adjacent points in time are dependent.

Goals

- Descriptive - Describe the observed series. More to unpack here than the mean, variance, and distribution.
- Inferential - Seek to determine from one time series data set what other data could have been observed or are expected to be observed in the future. To do this, we need a model.
- Predictive - Use the correlation in the series to predict the future.
- Control - Use information in the series to determine if the process generating the data is in control or not; and if not, adjust the process.

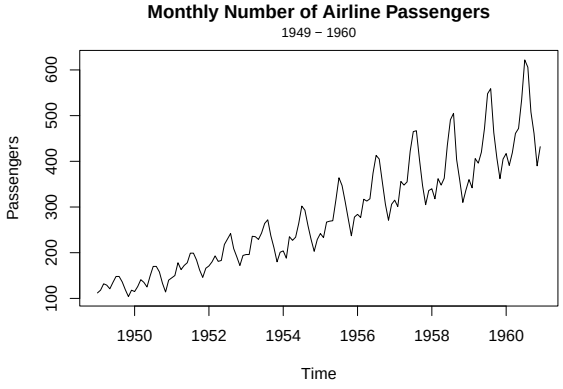
Considerations

- Nature of the time index?
- Is the process discrete or continuous?
- Univariate or multivariate?

First step: Look at Time Series Plot

- Time series plots will always have our variables of interest on the y-axis and time on the x-axis.
- What to look for:
  - Trend
  - Mean
  - Variability
  - Smoothness
  - Do these stay constant across our domain or do they change?

Example


Monthly Number of Airline Passengers
1949 – 1960
Passengers
100
200
300
400
500
600
1950
1952
1954
1956
1958
1960
Time

Gaussian AR(p) Process

- $\{W_t : t \in \mathbb{Z}\}$ is a Gaussian autoregressive process of order p, AR(p), if it can be represented as

$$W_t = \phi_1 W_{t-1} + \cdots + \phi_p W_{t-p} + \varepsilon_t,$$

  where ε_t are IID normal random variables mean $\mu_\varepsilon = 0$ and variance $\sigma_\varepsilon^2 < \infty$, independent of the $\{W_s : s < t\}$.

- Assuming the process is second-order stationary, the autocovariance function of W_t satisfies the Yule-Walker equations. In particular, for the AR(1) with $\phi = \phi_1$, the marginal mean and variance of W_t are

$$\mathrm{E}[W_t] = \frac{\mu_\varepsilon}{1-\phi} \quad \text{and} \quad \mathrm{Var}(W_t) = \frac{\sigma_\varepsilon^2}{1-\phi^2}.$$

Gaussian AR(1) Process

Let $\mathbf{W}_n = (W_n, \ldots, W_1)$. Then for $W_n = w$, the k-step ahead forecast is the k-step ahead conditional mean given by

$$\mathrm{E}\left[W_{n+k} \,|\, \mathbf{W}_n, W_n = w\right] = \phi^k w,$$

the k-step ahead conditional distribution is normal,

$$W_{n+k} \,|\, \mathbf{W}_n \sim \mathrm{N}\left(\mu_{W_{n+k}\,|\,\mathbf{W}_n}, \sigma^2_{W_{n+k}\,|\,\mathbf{W}_n}\right), \tag{1}$$

where for $W_n = w$

$$\mu_{W_{n+k}\,|\,\mathbf{W}_n} = \phi^k w + \mu_\varepsilon \frac{1-\phi^k}{1-\phi}, \tag{2}$$

$$\sigma^2_{W_{n+k}\,|\,\mathbf{W}_n} = \left(\frac{1-\phi^{2k}}{1-\phi^2}\right)\sigma^2_\varepsilon. \tag{3}$$

Motivation

- Discrete-valued time series occur in most of areas of scientific inquiry. But they're often modeled and forecasts are found using a continuous Gaussian process.
- In a series of seminal papers (Jacobs and Lewis, 1978a,b, 1983), Jacobs and Lewis introduced the theory and analysis for discrete time series processes (DARMA).
- A collection of discrete time series models that addresses some of the disadvantages of Jacobs and Lewis incorporates binomial thinning.

Binomial Thinning

Let B_j be IID Bernoulli(α) where for each j, $P(B_j = 1) = \alpha$. The binomial thinning operator $\circ$ is

$$\alpha \circ X = \sum_{j=1}^{X} B_j. \tag{4}$$

(Steutel and van Harn, 1979).

- If X is a process with Poisson marginals, the model is a first-order Poisson integer-valued autoregressive (INAR(1)) model (Al-Osh and Alzaid, 1987; McKenzie, 1988).
- If X has geometric marginals, the resultant model is a first-order geometric integer-values AR(1).
- Other approaches to address properties such as overdispersion or zero-inflation by utilizing the negative binomial, Poisson-Lindley, or zero-inflated Poisson.

Forecasting Discrete Processes

- A coherent forecast is one that does not violate known constraints.
- When traditional methods based on a Gaussian AR(p) model are used for forecasting, the forecast is not coherent.
- Most common procedure for constructing forecasts is to use conditional expectations.
- Consider a realization $\mathbf{X}_n = (X_n, \ldots, X_1)$ from a discrete time series. Two popular loss functions can be considered; each has a different k-step ahead forecast.
  - To minimize the k-step ahead conditional mean square error, the k-step ahead conditional mean is used for forecasting.
  - The k-step ahead conditional median minimizes the k-step ahead conditional mean absolute error.
  - The k-step ahead conditional mode can also be used effectively in forecasting discrete time series.

Geometric Integer Autoregressive Process

The process $\{X_t\}$ is a geometric integer-valued autoregressive process of order one, or GINAR(1) if it can be represented as

$$X_t = \alpha \circ X_{t-1} + \epsilon_t, \qquad t = 1, 2, \ldots, \quad 0 \leq \alpha \leq 1,$$

- The operator $\circ$ indicates binomial thinning defined in (4).
- $X_0 \sim$ geometric(θ) with $P(X_0 = 0) = \theta$.
- The innovation process $\{\epsilon_t\}$ is defined by $\epsilon_t = U_t W_t$ for all t, where U_t is independent of W_t and X_{t-1}, $U_t \sim$ IID Bernoulli$(1 - \alpha)$ and W_t are IID geometric(θ) (McKenzie, 1986).

GINAR(1)

- The PMF of ϵ_t is

$$P(\epsilon_t = z) = \begin{cases} 1 - \theta + \alpha\theta, & \text{for } z = 0, \\ (1-\theta)(1-\alpha)\theta^z, & \text{for } z = 1, 2, \ldots. \end{cases}$$

- It follows that for every t, the marginal distribution of X_t is also geometric(θ) having marginal mean and variance

$$\mathrm{E}[X_t] = \frac{\theta}{1-\theta} \quad \text{and} \quad \mathrm{Var}(X_t) = \frac{\theta}{(1-\theta)^2}.$$

Forecasting a GINAR Process

Awale et al. (2017) present an approach that produces k-step ahead forecasts based on a conditional GINAR(1) PMF. Let $P(X_{n+k} = y \,|\, \mathbf{X}_n, X_n = x)$ be written $p(y \,|\, x)$. Then

$$p(y \,|\, x) = \begin{cases} (1-\theta)\theta^{y-x} \sum_{j=0}^{y} \left\{ \binom{x}{j} \theta^{x-j} \alpha^j (1-\alpha)^{x-j+1} \right\} \\ \qquad + \binom{x}{y} \alpha^{y+1} (1-\alpha)^{x-y}, & \text{if } y \leq x, \\ (1-\theta)\theta^{y-x}(1-\alpha)\left\{\alpha + (1-\alpha)\theta\right\}^x, & \text{if } y > x. \end{cases}$$

Forecasting a GINAR Process

At time n, the k-step ahead conditional mean and conditional variance are

$$\begin{aligned}
\mathrm{E}\left[X_{n+k} \mid x\right] &= \alpha^k x + \frac{\theta(1-\alpha^k)}{1-\theta} \\
\mathrm{Var}\left(X_{n+k} \mid x\right) &= \alpha^k(1-\alpha^k)x \\
&\quad + \left(\frac{\alpha}{1-\alpha^2}\right)\left(1-\alpha^{k-1}-\alpha^k+\alpha^{2k-1}\right)\mu_\epsilon \\
&\quad + \left(\frac{1-\alpha^{2k}}{1-\alpha^2}\right)\sigma_\epsilon^2,
\end{aligned}$$

where μ_ϵ and σ_ϵ^2 are the mean and variance of ϵ_t given by

$$\mu_\epsilon = \frac{\theta(1-\alpha)}{1-\theta} \quad \text{and} \quad \sigma_\epsilon^2 = \frac{(1-\alpha)\theta(1+\alpha\theta)}{(1-\theta)^2}.$$

As $k \to \infty$, $\mathrm{E}\left[X_{n+k} \mid x\right] \to \mathrm{E}\left[X_n\right]$ and $\mathrm{Var}\left(X_{n+k} \mid x\right) \to \mathrm{Var}\left(X_n\right)$.

Poisson-Lindley Distribution

The Poisson-Lindley (PL) distribution was introduced by Sankaran (1970) to model over-dispersed count data.

- If $X \sim \text{PL}(\theta)$, the PMF of X is

$$P(X = x) = \frac{\theta^2(x + \theta + 2)}{(1 + \theta)^{x+3}}, \qquad x = 0, 1, 2, \ldots, \quad \theta > 0.$$

- The mean and variance of the $\text{PL}(\theta)$ distribution are

$$\text{E}[X] = \frac{\theta + 2}{\theta(\theta + 1)} \quad \text{and} \quad \text{Var}(X) = \frac{\theta^2 + 4\theta^2 + 6\theta + 2}{\theta^2(\theta + 1)^2}.$$

- In time series, Mohammadpour et al. (2018) introduced the INAR(1) process with PL marginals.
- One downside to the this model is that it can be complicated and difficult to estimate the model parameters.

Poisson-Lindley Integer Autoregressive Process

The process $\{X_t\}$ follows an order-one Poisson-Lindley autoregressive proces (PLINAR(1)) if

$$X_t = \alpha \circ X_{t-1} + Z_t, \qquad t = 1, 2, \ldots, \quad 0 \leq \alpha \leq 1,$$

- $Z_t = I_t H_t$ for all t. $I_t \sim$ Bernoulli$(1-\alpha)$ for all t. H_t is defined on $h = 0, 1, \ldots,$ having PMF

$$\begin{aligned} P(H_t = h) = & \frac{\theta^2(1-\alpha)^2 + \theta(1-\alpha^2) + 2\alpha}{\{\theta(1-\alpha)+1\}^2} \frac{\theta}{1+\theta}\left(1 - \frac{\theta}{1+\theta}\right) \\ & + \frac{(1-\alpha)}{\theta(1-\alpha)+1}(h+1)\left(\frac{\theta}{1+\theta}\right)^2\left(1 - \frac{\theta}{1+\theta}\right)^h \\ & - \frac{\alpha}{\{\theta(1-\alpha)+1\}^2}\frac{\theta+1}{\theta+1+\alpha}\left(1 - \frac{\theta+1}{\theta+1+\alpha}\right)^h. \end{aligned}$$

- $X_0 \sim$ PL(θ), independent of I_t and H_t.

Forecasting a PLINAR Process

The k-step ahead conditional PMF for the PLINAR(1), given by Mohammadpour et al. (2018) is

$$p(y \mid x) = \sum_{j=0}^{\min(y,x)} \binom{x}{j} \alpha^{kj} (1-\alpha^k)^{x-j} P(Z_{n+k} = y - j),$$

where

$$p(z) = \begin{cases} \alpha^k + (1-\alpha^k)\left\{A_k \frac{\theta}{1+\theta} + B_k \left(\frac{\theta}{1+\theta}\right)^2 + C_k \frac{1+\theta}{1+\theta+\alpha^k}\right\} & \text{if } z = 0, \\ (1-\alpha^k)\left\{A_k \frac{\theta}{1+\theta}\left(\frac{1}{1+\theta}\right)^z + B_k(z+1)\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{1}{1+\theta}\right)^z \right. & \\ \qquad \left. + C_k \frac{1+\theta}{1+\theta+\alpha^k}\left(\frac{\alpha^k}{1+\theta+\alpha^k}\right)^z\right\} & \text{if } z = 1, 2, \ldots, \end{cases}$$

and

$$A_k = \frac{\theta^2(1-\alpha^k)^2 + \theta(1-\alpha^{2k}) + 2\alpha^k}{\{\theta(1-\alpha^k)+1\}^2},$$

$$B_k = \frac{1-\alpha^k}{\theta(1-\alpha^k)+1}, \quad C_k = \frac{-\alpha^k}{\{\theta(1-\alpha^k)+1\}^2}.$$

Forecasting a PLINAR Process

The k-step ahead conditional mean and variance provided by Mohammadpour et al. (2018) are, respectively

$$\mathrm{E}\left[X_{n+k} \mid x\right] = \alpha^k x + (1-\alpha^k)\frac{\theta+2}{\theta(\theta+1)}$$

$$\mathrm{Var}\left(X_{n+k} \mid x\right) = \alpha^k(1-\alpha^k)x + \left(\frac{1-\alpha^{2k}}{1-\alpha^2}\right)\sigma_\epsilon^2 + \left\{\frac{(1-\alpha^k)(\alpha-\alpha^{k+1})}{1-\alpha^2}\right\}\mu_\epsilon,$$

where μ_ϵ and σ_ϵ^2 are the mean and variance of ϵ_t given by

$$\mu_\epsilon = \frac{(1-\alpha)(\theta+2)}{\theta(\theta+1)}$$

$$\sigma_\epsilon^2 = \frac{(1-\alpha)\left\{\theta^3+4\theta^2+6\theta+2+\alpha(\theta^2+4\theta+2)\right\}}{\theta^2(\theta+1)^2}.$$

As $k \to \infty, \mathrm{E}\left[X_{n+k} \mid x\right] \to \mathrm{E}\left[X_n\right]$ and $\mathrm{Var}\left(X_{n+k} \mid x\right) \to \mathrm{Var}\left(X_n\right)$.

Zero-Inflated Poisson Distribution

If the data contains a large number of zeros, then a model to address over-dispersion, e.g., the Poisson-Lindley model, is not enough.

- Let X be a zero-inflated Poisson(λ, π) random variable. Then the PMF of X is

$$P(X = x) = \begin{cases} (1-\pi) + \pi e^{-\lambda} & \text{for } x = 0, \\ \pi \left(\frac{e^{-\lambda}\lambda^x}{x!} \right), & \text{for } x = 1, 2, \ldots, \end{cases} \tag{5}$$

  where $0 < \pi < 1$ and $\lambda > 0$.

- The mean and variance of X is

$$\mathrm{E}[X_t] = \pi\lambda \quad \text{and} \quad \mathrm{Var}(X_t) = \pi\lambda + \pi(1-\pi)\lambda^2.$$

Zero-inflated Poisson Integer Autoregressive Process

The process $\{X_t\}$ is a ZIPINAR(1) if it can be represented as

$$X_t = Y_t Z_t \qquad t = 0, 1, 2, \ldots,$$

- $\{Y_t\}$ is the Poisson INAR(1) defined by

$$Y_t = \alpha \circ Y_{t-1} + \epsilon_t, \qquad t = 1, 2, \ldots, \quad 0 \leq \alpha \leq 1,$$

- $Y_0 \sim$ Poisson(λ)
- $\{\epsilon_t\}$ being IID Poisson($\lambda(1-\alpha)$).
- $\{Z_t\}$ is an IID Bernoulli(π), i.e. $P(Z_t = 1) = \pi$.
- See Al-Osh and Alzaid (1987); McKenzie (1988); Maiti et al. (2015).

The ZIPINAR(1), k-step-ahead conditional PMF is given by

$$p(y \mid x) = \begin{cases} (1-\pi) + \frac{\pi e^{-\lambda}}{(1-\pi)+\pi e^{-\lambda}} \\ \quad \times \left\{(1-\pi) + \pi e^{-\lambda(1-\alpha^k)}\right\}, & \text{if } y \neq 0, x = 0, \\ \\ \frac{\pi}{(1-\pi)+\pi e^{-\lambda}} \left\{(1-\pi)\left(\frac{e^{-\lambda}\lambda^y}{y!}\right) \right. \\ \quad \left. + \pi e^{-\lambda}\left(\frac{e^{-\lambda(1-\alpha^k)}\left\{\lambda(1-\alpha^k)\right\}^y}{y!}\right)\right\}, & \text{if } y = 0, x \neq 0, \\ \\ (1-\pi) + \pi(1-\alpha^k)^x e^{-\lambda(1-\alpha^k)}, & \text{if } y \neq 0, x \neq 0, \\ \\ \pi \sum_{j=0}^{\min(y,x)} \frac{1}{(y-j)!}\left[\binom{x}{j}(\alpha^k)^j(1-\alpha^k)^{x-j} \right. \\ \quad \left. e^{-\lambda(1-\alpha^k)}\left\{\lambda(1-\alpha^k)\right\}^{y-j}\right], & \text{if } y = x = 0, \end{cases} \tag{6}$$

At time n, the k-step-ahead conditional mean and conditional variance are

$$\begin{aligned}
\mathrm{E}\left[X_{n+k} \mid x\right] &= \pi\lambda\left(1 - \frac{\pi e^{-\lambda}\alpha^k}{1-\pi+\pi e^{-\lambda}}\right) I(X_n = 0) \\
&\quad + \left\{\pi\lambda(1-\alpha^k) + \pi\alpha^k X_n\right\} I(X_n > 0) \qquad (7) \\
\mathrm{Var}\left(X_{n+k} \mid x\right) &= \pi\lambda\left(1 - \frac{\pi e^{-\lambda}\alpha^k}{1-\pi+\pi e^{-\lambda}}\right) I(X_n = 0) \\
&\quad + \left\{\pi\lambda(1-\alpha^k) + \pi\alpha^k X_n^2\right\} I(X_n > 0) \\
&\quad - \left[\pi\lambda\left(1 - \frac{\pi e^{-\lambda}\alpha^k}{1-\pi+\pi e^{-\lambda}}\right) I(X_n = 0)\right. \\
&\quad \left. + \left\{\pi\lambda(1-\alpha^k) + \pi\alpha^k X_n\right\} I(X_n > 0)\right]^2
\end{aligned}$$

As $k \to \infty$, $\mathrm{E}\left[X_{n+k} \mid x\right] \to \mathrm{E}\left[X_n\right]$ and $\mathrm{Var}\left(X_{n+k} \mid x\right) \to \mathrm{Var}\left(X_n\right)$.



Comparing Forecasts: Background

- In Maiti et al. (2016), various INAR(1) processes were compared to the Gaussian autoregressive process, where the forecast for the Gaussian model was rounded to the nearest integer. They found the methods produced similar forecasts.
- In Alwan and Weiß (2016) the marginal forecast distribution of INAR(1) processes was approximated by the Gaussian AR(1) conditional forecast distribution, and it was found that the approximation was worse, especially in the range of low counts data.
- Bisaglia and Gerolimetto (2015) showed when systematically comparing a coherent point forecast method of an INAR(1) model to the approximate point forecast obtained from a Gaussian AR(1), the Gaussian had higher forecast mean square error and forecast mean absolute error.
- Homburg (2020) thoroughly investigated the differences between the Poisson INAR(1) to the Gaussian AR(1) using a variety of measures to compare the forecasts from the competing models. Her main conclusion is the quality of the approximation is influenced by the amount of autocorrelation present.



Parameter Estimation for ZIPINAR(1)

- Three techniques are prevalent in the literature for estimating the parameters of the ZIPINAR(1) model
  - conditional least squares,
  - Yule-Walker, and
  - maximum likelihood.
- For the ZIPINAR(1) model, little has been done to evaluate how well the different methods perform.
- There have been no studies on how parameter estimation effects forecast efficacy.

Conditional Least Squares Estimators

Conditional least squares (CLS) estimators are found by minimizing

$$Q = \sum_{t=2}^{n} \{X_t - \mathrm{E}[X_t | X_{t-1}]\}^2,$$

where $\mathrm{E}[X_t|X_{t-1}]$ is (7) when $k = 1$ (see, for example, Maiti et al. (2015)).

- A closed-form expression for α, π, and λ is not possible. Consequently, values of α, π, and λ that minimize Q must be found numerically.
- The optimization algorithm of Byrd et al. (1995) which allows that each parameter is given an lower and upper bound was implemented via the `optim` function in *R* using the option `method = "L-BFGS-B"`.

Yule-Walker Estimates

- The Yule-Walker (YW) estimators are the result of a type of method-of-moments approach for finding expressions for estimators of α, π and λ.
- For the ZIPINAR(1) model, the Yule-Walker estimators are

$$\hat{\alpha}_{YW} = \frac{\left\{\sum_{t=1}^{n-1}(X_t - \bar{X})(X_{t+1} - \bar{X})\right\}\left\{\sum_{t=1}^{n}(X_t^2 - \bar{X})\right\}}{n^2\bar{X}^3},$$

$$\hat{\pi}_{YW} = \frac{n\bar{X}^2}{\sum_{t=1}^{n}(X_t^2 - \bar{X})}, \qquad \hat{\lambda}_{YW} = \frac{\sum_{t=1}^{n}(X_t^2 - \bar{X})}{n\bar{X}}.$$

- In some cases, conditional least squares and Yule-Walker estimators are equivalent.

Maximum Likelihood Estimates

Finally, the maximum likelihood (ML) estimators of α, π and λ are the values $\hat{\alpha}_{ML}, \hat{\pi}_{ML}$ and $\hat{\lambda}_{ML}$ that maximize the log-likelihood given by

$$\ln L(X_1, \ldots, X_n; \alpha, \pi, \lambda) = p(X_1) \prod_{t=2}^{n} \ln p(X_t | X_{t-1})$$

where $p(X_1)$ is found in (5) and $p(X_t|X_{t-1})$ is (6) when $k = 1$. Like conditional least squares, values for the MLEs are found numerically, using the same optimization algorithm.

Simulation Design

- Maiti et al. (2015) presented an extremely limited simulation study to compare the performance of the three estimation methods for the ZIPINAR(1) parameters, considering only $\alpha = 0.6, \lambda = 1.5$, and $\pi = 0.4$ or 0.8.
- We investigate the performances of all three methods of estimation for a more thorough set of values of all the parameters.
  1. Generate 500 replicates of a ZIPINAR(1) for series lengths $n = 100, 500, 1000,$ and 5000 for all combinations of $\alpha = 0.2, 0.4, 0.6,$ and 0.8, $\pi = 0.2, 0.4, 0.6,$ and 0.8, and $\lambda = 1, 2, 3$.
  2. For each series, compute estimates of all parameters using all three estimation methods.
  3. Record iterations that result values that converge to the boundaries (CLS) or are inadmissible (YW).
  4. Compute the averages and standard errors of all admissible values for CLS, YW, and ML.

Simulation

n	$\hat{\alpha}_{CLS}$	$\hat{\pi}_{CLS}$	$\hat{\lambda}_{CLS}$	$\hat{\alpha}_{YW}$	$\hat{\pi}_{YW}$	$\hat{\lambda}_{YW}$	$\hat{\alpha}_{ML}$	$\hat{\pi}_{ML}$	$\hat{\lambda}_{ML}$
$\alpha = 0.2, \pi = 0.2, \lambda = 1$									
100	0.2894	0.1633	1.9062	0.3746	0.2372	0.9676	0.2027	0.2255	0.9843
SE	0.2801	0.0838	2.5768	0.2819	0.1034	0.4115	0.3334	0.0931	0.3622
500	0.3509	0.1634	1.3416	0.3786	0.2078	0.9831	0.2143	0.2043	0.9918
SE	0.3011	0.0756	1.3152	0.2556	0.0359	0.1820	0.2485	0.0314	0.1631
1000	0.3141	0.1680	1.3130	0.3335	0.2038	0.9929	0.1987	0.2028	0.9945
SE	0.2890	0.1011	1.4228	0.2391	0.0241	0.1219	0.2055	0.0217	0.1143
5000	0.2771	0.1940	1.1576	0.2220	0.2007	0.9998	0.1918	0.2003	1.0014
SE	0.2208	0.1189	0.9968	0.1343	0.0105	0.0545	0.1190	0.0095	0.0505
$\alpha = 0.2, \pi = 0.2, \lambda = 2$									
100	0.4585	0.1774	3.4582	0.4220	0.2102	1.9545	0.3120	0.2033	1.9997
SE	0.3040	0.1124	3.1764	0.2649	0.0519	0.4612	0.3610	0.0455	0.4061
500	0.3963	0.1855	2.7755	0.4207	0.2015	1.9762	0.2366	0.2004	1.9824
SE	0.2842	0.1146	2.5203	0.2646	0.0223	0.2019	0.2335	0.0202	0.1841
1000	0.3372	0.1909	2.7084	0.3822	0.2006	1.9943	0.2164	0.1995	2.0008
SE	0.2470	0.1326	2.4189	0.2616	0.0166	0.1513	0.1815	0.0138	0.1248
5000	0.2394	0.2137	2.2130	0.2810	0.2004	1.9999	0.2050	0.2003	1.9994
SE	0.1609	0.1115	1.5678	0.1683	0.0074	0.0627	0.0928	0.0066	0.0557
$\alpha = 0.2, \pi = 0.2, \lambda = 3$									
100	0.4743	0.1884	4.3647	0.4689	0.2045	2.9582	0.3133	0.2015	2.9918
SE	0.3071	0.1218	3.1936	0.2660	0.0422	0.4870	0.3348	0.0392	0.4314
500	0.4103	0.1837	4.0133	0.3993	0.2020	2.9840	0.2196	0.2004	3.0039
SE	0.2621	0.0989	2.7315	0.2795	0.0205	0.2352	0.2014	0.0186	0.1970
1000	0.3372	0.1987	3.6469	0.4069	0.2000	2.9886	0.2061	0.1996	2.9917
SE	0.2522	0.1285	2.5080	0.2504	0.0137	0.1607	0.1542	0.0123	0.1335
5000	0.2305	0.2185	3.3580	0.3017	0.1999	3.0031	0.1995	0.1999	3.0023
SE	0.1423	0.1283	1.9146	0.2033	0.0064	0.0723	0.0806	0.0058	0.0601

Simulation

n	$\hat{\alpha}_{\text{CLS}} < 0$	$\hat{\alpha}_{\text{CLS}} > 1$	$\hat{\pi}_{\text{CLS}} < 0$	$\hat{\pi}_{\text{CLS}} > 1$	$\hat{\alpha}_{\text{YW}} < 0$	$\hat{\alpha}_{\text{YW}} > 1$	$\hat{\pi}_{\text{YW}} < 0$	$\hat{\pi}_{\text{YW}} > 1$
$\alpha = 0.2, \pi = 0.2, \lambda = 1$								
100	0.512	0.266	0.000	0.132	0.600	0.104	0.000	0.000
500	0.334	0.174	0.000	0.182	0.382	0.048	0.000	0.000
1000	0.224	0.120	0.000	0.226	0.268	0.014	0.000	0.000
5000	0.088	0.004	0.000	0.162	0.096	0.000	0.000	0.000
$\alpha = 0.2, \pi = 0.2, \lambda = 2$								
100	0.370	0.268	0.000	0.102	0.580	0.200	0.000	0.000
500	0.284	0.096	0.000	0.136	0.406	0.102	0.000	0.000
1000	0.180	0.022	0.000	0.154	0.326	0.054	0.000	0.000
5000	0.044	0.000	0.000	0.094	0.154	0.000	0.000	0.000
$\alpha = 0.2, \pi = 0.2, \lambda = 3$								
100	0.336	0.288	0.000	0.094	0.566	0.264	0.000	0.000
500	0.268	0.060	0.000	0.098	0.440	0.146	0.000	0.000
1000	0.200	0.010	0.000	0.124	0.398	0.084	0.000	0.000
5000	0.064	0.000	0.000	0.054	0.226	0.002	0.000	0.000

Simulation

n	$\hat{\alpha}_{CLS}$	$\hat{\pi}_{CLS}$	$\hat{\lambda}_{CLS}$	$\hat{\alpha}_{YW}$	$\hat{\pi}_{YW}$	$\hat{\lambda}_{YW}$	$\hat{\alpha}_{ML}$	$\hat{\pi}_{ML}$	$\hat{\lambda}_{ML}$
$\alpha = 0.8, \pi = 0.4, \lambda = 1$									
100	0.4383	0.3848	1.3753	0.4302	0.4722	0.9491	0.7098	0.4364	1.0042
SE	0.2860	0.1727	1.7403	0.2704	0.1575	0.4287	0.2540	0.1387	0.3798
500	0.5050	0.4448	0.8795	0.5921	0.4205	0.9638	0.7777	0.4068	0.9876
SE	0.2789	0.1241	0.3408	0.2209	0.0646	0.1978	0.0854	0.0454	0.1637
1000	0.5927	0.4446	0.9090	0.6761	0.4100	0.9856	0.7899	0.4030	0.9942
SE	0.2410	0.1015	0.2222	0.1889	0.0488	0.1459	0.0458	0.0315	0.1130
5000	0.7586	0.4094	0.9875	0.7691	0.4032	0.9947	0.8000	0.4013	0.9981
SE	0.1468	0.0415	0.1031	0.1124	0.0212	0.0683	0.0181	0.0136	0.0512
$\alpha = 0.8, \pi = 0.4, \lambda = 2$									
100	0.5080	0.3865	2.8817	0.4624	0.4313	1.9194	0.7562	0.4101	1.9843
SE	0.2613	0.1725	2.5697	0.2740	0.0901	0.5294	0.1645	0.0660	0.4730
500	0.6542	0.4079	2.0595	0.5771	0.4048	1.9825	0.7935	0.4006	1.9991
SE	0.2200	0.0868	0.6147	0.2394	0.0361	0.2545	0.0466	0.0269	0.2157
1000	0.7237	0.4041	2.0195	0.6535	0.4041	1.9960	0.7976	0.4015	2.0060
SE	0.1560	0.0468	0.2770	0.2010	0.0260	0.1872	0.0299	0.0192	0.1476
5000	0.7960	0.3994	2.0093	0.7701	0.4016	1.9935	0.7995	0.4008	1.9979
SE	0.0819	0.0166	0.1039	0.1114	0.0123	0.0783	0.0129	0.0092	0.0669
$\alpha = 0.8, \pi = 0.4, \lambda = 3$									
100	0.5565	0.3921	3.8999	0.4388	0.4203	2.9421	0.7866	0.4046	3.0289
SE	0.2591	0.1554	2.4673	0.2769	0.0671	0.6127	0.0912	0.0540	0.5672
500	0.7145	0.3967	3.1468	0.5518	0.4034	2.9759	0.7949	0.4009	2.9867
SE	0.1874	0.0664	0.7880	0.2506	0.0317	0.2993	0.0381	0.0250	0.2568
1000	0.7373	0.3990	3.0482	0.6216	0.4024	2.9896	0.7964	0.4001	3.0056
SE	0.1391	0.0409	0.3970	0.2205	0.0215	0.2154	0.0258	0.0167	0.1827
5000	0.7960	0.3994	3.0120	0.7654	0.4006	2.9965	0.8003	0.4002	3.0000
SE	0.0766	0.0152	0.1557	0.1214	0.0096	0.0910	0.0116	0.0074	0.0769

Simulation

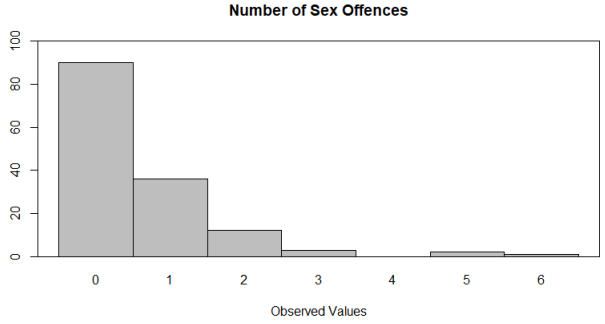
n	$\hat{\alpha}_{\text{CLS}} < 0$	$\hat{\alpha}_{\text{CLS}} > 1$	$\hat{\pi}_{\text{CLS}} < 0$	$\hat{\pi}_{\text{CLS}} > 1$	$\hat{\alpha}_{\text{YW}} < 0$	$\hat{\alpha}_{\text{YW}} > 1$	$\hat{\pi}_{\text{YW}} < 0$	$\hat{\pi}_{\text{YW}} > 1$
$\alpha = 0.8, \pi = 0.4, \lambda = 1$								
100	0.112	0.346	0.000	0.198	0.132	0.234	0.000	0.020
500	0.008	0.264	0.000	0.152	0.002	0.194	0.000	0.000
1000	0.000	0.250	0.000	0.054	0.000	0.200	0.000	0.000
5000	0.000	0.124	0.000	0.000	0.000	0.086	0.000	0.000
$\alpha = 0.8, \pi = 0.4, \lambda = 2$								
100	0.058	0.320	0.000	0.074	0.222	0.284	0.000	0.000
500	0.002	0.224	0.000	0.002	0.018	0.258	0.000	0.000
1000	0.000	0.124	0.000	0.000	0.002	0.204	0.000	0.000
5000	0.000	0.008	0.000	0.000	0.000	0.056	0.000	0.000
$\alpha = 0.8, \pi = 0.4, \lambda = 3$								
100	0.048	0.322	0.000	0.038	0.258	0.312	0.000	0.000
500	0.000	0.194	0.000	0.000	0.040	0.280	0.000	0.000
1000	0.000	0.098	0.000	0.000	0.004	0.232	0.000	0.000
5000	0.000	0.006	0.000	0.000	0.000	0.086	0.000	0.000

Application: Sex Offences

Below is 144 counts of sex offences reported in the 21st police car beat in Pittsburgh for one month, beginning in Jan. 1990 through Dec. 2001 (Ristic et al., 2009). The data is available in the *R* library `tsinteger` under the name `sexoffences`.

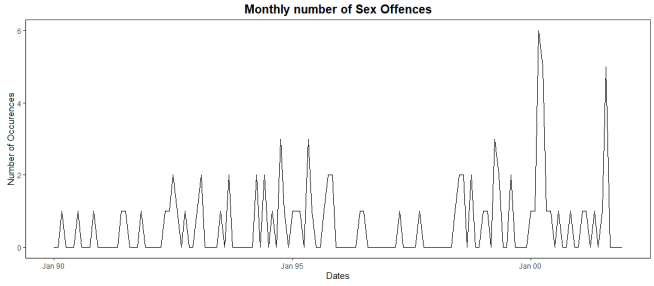
	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1990	0	0	1	0	0	0	1	0	0	0	1	0
1991	0	0	0	0	0	1	1	0	0	0	1	0
1992	0	0	0	0	1	1	2	1	0	1	0	0
1993	1	2	0	0	0	0	1	0	2	0	0	0
1994	0	0	0	2	0	2	0	1	0	3	1	0
1995	1	1	1	0	3	1	0	0	1	2	2	0
1996	0	0	0	0	0	1	1	0	0	0	0	0
1997	0	0	0	1	0	0	0	0	1	0	0	0
1998	0	0	0	0	0	1	2	2	0	2	0	0
1999	1	1	0	3	2	0	0	2	0	0	0	0
2000	1	1	6	5	1	1	0	1	0	0	1	0
2001	0	1	1	0	1	0	1	5	0	0	0	0

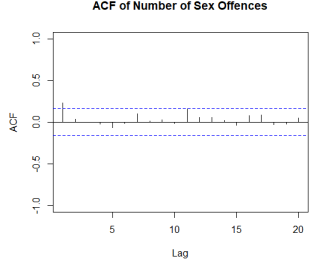
Descriptive Measures of Sex Offences Data

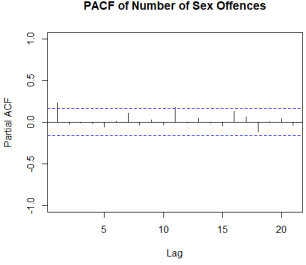


Minimum	Maximum	Median	Mode	Mean	Variance
0	6	0	0	0.5903	1.0268

Descriptive Measures of Sex Offences Data







- The index of dispersion is $\hat{I} = s_X^2/\bar{X} = 1.7395$, which is large enough to conclude that the data is over-dispersed.
- Ninety of the 144 (62.5%) data values are zero.
- The bottom two graphs on the previous slide are the sample ACF (left) and sample PACF (right) with large sample 95% confidence bounds for white noise. Both indicate an autoregressive structure of order one is appropriate.
- Akaike information criteria (AIC), corrected AIC (AICc) and the Bayesian information criteria (BIC) are also used for order determination. All three indicate order one.

	AIC	AICc	BIC
AR(1)	**409.3019**	**409.4734**	**418.2114**
AR(2)	411.2420	411.5298	423.1212
AR(3)	413.2324	413.6671	428.0814

Parameter Estimates for ZIP-INAR(1) Model Parameters

The table below provides estimates for α, π and λ using the conditional least squares (CLS), Yule-Walker (YW) and maximum likelihood (ML) when fitting the ZIPINAR(1) model.

	CLS	YW	ML
α	0.3958	0.8588	0.2195
π	0.6922	0.4575	0.6446
λ	0.8729	1.3176	0.9190

ZIPINAR(1) k-Step Ahead Conditional Distribution

Estimated conditional k-step-ahead probability mass function for $X_n = 0$ for three estimation methods for α, π and λ based on ZIPINAR(1) model for the sex offences data.

	CLS				YW				ML			
	$k=1$	$k=2$	$k=3$	$k=\infty$	$k=1$	$k=2$	$k=3$	$k=\infty$	$k=1$	$k=2$	$k=3$	$k=\infty$
$p_k(0\|0)$	0.655	0.617	0.605	0.597	0.712	0.702	0.694	0.665	0.637	0.617	0.614	0.613
$p_k(1\|0)$	0.235	0.248	0.251	0.252	0.145	0.152	0.157	0.161	0.232	0.236	0.236	0.236
$p_k(2\|0)$	0.084	0.100	0.106	0.110	0.088	0.090	0.093	0.106	0.097	0.106	0.108	0.109
$p_k(3\|0)$	0.021	0.027	0.030	0.032	0.038	0.039	0.039	0.047	0.027	0.032	0.033	0.033
$p_k(4\|0)$	0.004	0.006	0.006	0.007	0.013	0.013	0.013	0.015	0.006	0.007	0.008	0.008
$p_k(5\|0)$	0.001	0.001	0.001	0.001	0.003	0.003	0.003	0.004	0.001	0.001	0.001	0.001
$p_k(6\|0)$	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000
$p_k(7\|0)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$p_k(8\|0)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

PLINAR(1) k-Step Ahead Conditional Distribution

Estimated conditional k-step-ahead probability mass function for $X_n = 0$ for three estimation methods for α and θ based on PLINAR(1) model for the sex offences data.

	CLS				YW				MLE			
	$k=1$	$k=2$	$k=3$	$k=\infty$	$k=1$	$k=2$	$k=3$	$k=\infty$	$k=1$	$k=2$	$k=3$	$k=\infty$
$p_k(0\|0)$	0.703	0.636	0.621	0.616	0.704	0.637	0.622	0.618	0.657	0.623	0.619	0.619
$p_k(1\|0)$	0.188	0.229	0.238	0.241	0.188	0.229	0.238	0.241	0.217	0.238	0.240	0.240
$p_k(2\|0)$	0.070	0.086	0.090	0.091	0.070	0.086	0.089	0.090	0.081	0.089	0.090	0.090
$p_k(3\|0)$	0.025	0.032	0.033	0.033	0.025	0.031	0.033	0.033	0.029	0.032	0.033	0.033
$p_k(4\|0)$	0.009	0.011	0.012	0.012	0.009	0.011	0.012	0.012	0.010	0.012	0.012	0.012
$p_k(5\|0)$	0.003	0.004	0.004	0.004	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004
$p_k(6\|0)$	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
$p_k(7\|0)$	0.000	0.000	0.001	0.001	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000
$p_k(8\|0)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

GINAR(1) k-Step Ahead Conditional Distribution

Estimated conditional k-step-ahead probability mass function for $X_n = 0$ for three estimation methods for α, and θ based on GINAR(1) model for the sex offences data.

	CLS				YW				ML			
	$k=1$	$k=2$	$k=3$	$k=\infty$	$k=1$	$k=2$	$k=3$	$k=\infty$	$k=1$	$k=2$	$k=3$	$k=\infty$
$p_k(0\|0)$	0.709	0.642	0.627	0.622	0.710	0.644	0.628	0.624	0.662	0.629	0.626	0.626
$p_k(1\|0)$	0.181	0.223	0.232	0.235	0.181	0.222	0.232	0.235	0.211	0.232	0.234	0.234
$p_k(2\|0)$	0.068	0.084	0.088	0.089	0.068	0.084	0.087	0.088	0.079	0.087	0.088	0.088
$p_k(3\|0)$	0.026	0.032	0.033	0.034	0.026	0.031	0.033	0.033	0.030	0.033	0.033	0.033
$p_k(4\|0)$	0.010	0.012	0.013	0.013	0.010	0.012	0.012	0.012	0.011	0.012	0.012	0.012
$p_k(5\|0)$	0.004	0.005	0.005	0.005	0.004	0.004	0.005	0.005	0.004	0.005	0.005	0.005
$p_k(6\|0)$	0.001	0.002	0.002	0.002	0.001	0.002	0.002	0.002	0.002	0.002	0.002	0.002
$p_k(7\|0)$	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
$p_k(8\|0)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Assessing Forecasts Performance

- To investigate the accuracy of the forecast, we partition an observed dataset of length n into two sets. The longer set, called the training set, contains the first m observations, and is used to estimate the parameters of the models. The remaining $n - m$ observations are used for evaluating forecasts using each of the following three criteria.
- For the sex offences data, the training set was the first 80% (115 observations) of the data. We then computed the k-step-ahead forecasts ($k = 1, 2, 3$) for the remaining 20% (29 observations).
- Four measures of forecast accuracy are computed on the forecasts.
  - Prediction root mean square error
  - Prediction mean absolute deviation
  - Percent true prediction for the median as the forecast
  - Percent true prediction for the mode as a forecast

k-Step Ahead Prediction Errors

- The k-step-ahead prediction root mean square error, PRMSE(k), is defined as

$$\text{PRMSE}(k) = \sqrt{\frac{1}{n-m-k+1}\sum_{t=m+k}^{n}\left(X_t - \hat{X}_t^{me}\right)^2},$$

where $\hat{X}_{t+k}^{me}$ is estimated k-step-ahead conditional mean given in equation (7).

- The k-step-ahead prediction mean absolute deviation, PMAD(k), is given by

$$\text{PMAD}(k) = \frac{1}{n-m-k+1}\sum_{t=m+k}^{n}\left|X_t - \hat{X}_t^{med}\right|,$$

where $\hat{X}_{t+k}^{med}$ is the median of the estimated k-step-ahead conditional distribution in equation (6).

k-Step Ahead Percent of True Prediction

- In the following equation, let $\hat{X}_t$ represent some k-step ahead forecast based on the estimated k-step ahead conditional distribution.
- Then, for the selected forecaster, the k-step ahead percentage of true prediction, PTP(k), is given by

$$\text{PTP}(k) = \frac{\sum_{t=m+k}^{n} I\left(X_t = \hat{X}_t\right)}{n - m - k + 1} \times 100\%,$$

where $I(\cdot)$ is an indicator function equaling one if $X_t = \hat{X}_t$ and zero otherwise.

- In our studies, we chose the median as one forecaster, and the mode as another.

Assessing Forecast Accuracy for Sex Offences Data

The table below contains the PRMSE, PMAD, and PTP for both the median and the mode.

		PRMSE			PMAD			PTP (Med.)			PTP (Mode)		
		$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$
CLS	GINAR(1)	1.597	1.669	1.700	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	PLINAR(1)	1.597	1.669	1.700	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	ZIPINAR(1)	1.857	1.852	1.886	0.966	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	Traditional	1.597	1.669	1.700	0.966	0.929	0.889	0.414	0.464	0.519	0.414	0.464	0.519
YW	GINAR(1)	1.597	1.669	1.895	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	PLINAR(1)	1.597	1.669	1.895	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	ZIPINAR(1)	1.857	1.852	1.886	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	Traditional	1.597	1.669	1.895	0.966	0.929	1.000	0.414	0.464	0.481	0.414	0.464	0.481
ML	GINAR(1)	1.597	1.852	1.886	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	PLINAR(1)	1.597	1.852	1.886	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	ZIPINAR(1)	1.857	1.852	1.886	0.931	0.929	0.963	0.517	0.536	0.519	0.517	0.536	0.519
	Traditional	1.597	1.669	1.895	0.966	0.929	1.000	0.414	0.464	0.481	0.414	0.464	0.481

Based on the results above, and relying more heavily on the PTP measures, the Discrete processes outperformed the traditional Gaussian model. For this data, the method of parameter estimation had little effect on the forecasts.

Assessing Forecast Accuracy for Skin Lesions Data

The table below contains the PRMSE, PMAD, and PTP for both the median and the mode.

		PRMSE			PMAD			PTP (Med.)			PTP (Mode)		
		$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$
CLS	GINAR(1)	2.744	3.021	3.120	1.765	2.000	2.133	0.294	0.188	0.133	0.353	0.312	0.333
	PLINAR(1)	2.744	3.021	3.120	1.765	2.000	2.133	0.294	0.188	0.133	0.353	0.312	0.333
	ZIPINAR(1)	-	-	-	-	-	-	-	-	-	-	-	-
	Traditional	2.744	3.021	3.120	1.765	2.000	2.133	0.235	0.188	0.133	0.235	0.188	0.133
YW	GINAR(1)	2.744	3.021	3.120	1.765	2.000	2.133	0.294	0.188	0.133	0.353	0.312	0.333
	PLINAR(1)	2.744	3.021	3.120	1.765	2.000	2.133	0.294	0.188	0.133	0.353	0.312	0.333
	ZIPINAR(1)	2.931	3.021	3.120	1.941	1.875	2.133	0.235	0.188	0.133	0.294	0.312	0.333
	Traditional	2.744	3.021	3.120	1.765	2.000	2.133	0.235	0.188	0.133	0.235	0.188	0.133
ML	GINAR(1)	2.931	3.021	3.120	1.882	2.000	2.133	0.235	0.188	0.133	0.294	0.312	0.333
	PLINAR(1)	2.910	3.021	3.120	1.882	2.000	2.133	0.235	0.188	0.133	0.294	0.312	0.333
	ZIPINAR(1)	2.931	3.021	3.120	1.882	2.000	2.133	0.235	0.188	0.133	0.294	0.312	0.333
	Traditional	2.744	3.021	3.120	1.765	2.000	2.133	0.235	0.188	0.133	0.235	0.188	0.133

Based on the results above, and relying more heavily on the PTP measures, the Discrete processes outperformed the traditional Gaussian model. For this data, the method of parameter estimation had little effect on the forecasts.

Conclusion/Future Research

- Create an R package for discrete-valued time series.
- Look into seasonal INAR(1).
- Investigate the performances of prediction intervals related to the topics considered.

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Thank You! Any Questions?